Efficient Data-Driven Abstraction of Monotone Systems with Disturbances

Anas Makdesi, Antoine Girard, Laurent Fribourg

Efficient Data-Driven Abstraction of Monotone Systems with Disturbances ⋆

Anas Makdesi * **Antoine Girard** * **Laurent Fribourg** **

* *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France {anas.makdesi,antoine.girard}@l2s.centralesupelec.fr*
** *Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France e-mail: fribourg@lsv.fr*

Abstract: In this paper, we present a novel approach for the abstraction of monotone systems with bounded disturbances. The approach is data-driven and uses a given set of samples of the (unknown) dynamics of the system to compute an abstraction defined on partitions of the state and input spaces. The proposed method is efficient as its computational complexity is linear in the number of samples and in the size of the partitions. Moreover, the abstraction is shown to be minimally conservative in the absence of disturbances. We show that the resulting symbolic model is itself a monotone transition system and is related to the original system by an alternating simulation relation. We present some numerical experiments to show the effectiveness of the approach and to show how the choice of the partitions or the number of samples affects the quality of the approximation.

Keywords: Monotone transition systems, data-driven control, symbolic control, abstraction.

1. INTRODUCTION

Symbolic control is a computational approach to controller synthesis, based on the use of discrete abstractions (a.k.a. symbolic models) of continuous dynamical systems and of discrete controller synthesis techniques (see e.g. Tabuada (2009); Belta et al. (2017)). The main advantages of symbolic control are as follows. Firstly, it can be applied to general classes of nonlinear systems with input and state constraints, as long as one is able to compute an over-approximation of the system's reachable sets, using e.g. approaches based on mixed-monotonicity (Coogan and Arcak (2017)) or on growth bounds (Reissig et al. (2016)). Secondly, since controller synthesis is handled using algorithmic approaches developed in the fields of supervisory control (Ramadge and Wonham (1987)) or formal methods (Bloem et al. (2012)), the type of specifications that can be addressed is broad: safety and reachability (Girard (2012)), behaviors described by automata (Pola and Di Benedetto (2019)), or temporal logic formulas (Belta et al. (2017)).

While symbolic control is usually presented as a model-driven technique, some approaches such as those for monotone systems (Meyer et al. (2015)) only require us to be able to sample the system dynamics on a given grid of states and inputs, and it seems natural to extend these techniques towards a purely data-driven approach. In this paper, we present a data-driven approach to abstraction of monotone systems with disturbances. We assume that we are given a set of samples of the dynamics of the system. Contrarily to model-driven approaches, the samples are assumed to be taken at random and given a priori. In addition to the data, we assume that we know bounds on the amplitude of disturbances affecting the system. Using only this information, we present an approach to compute a symbolic model of the system defined on partitions of the state and input spaces. The symbolic model and the original system are shown to be related by an alternating simulation relation (Tabuada (2009)) and therefore, the symbolic model can be used to synthesize controllers that are known to be correct by design. Moreover, we show that in the absence of disturbances, the abstraction is minimally conservative in the sense that it is alternatingly simulated by all monotone systems that are consistent with the data, but not more. In addition, we show that the symbolic model is itself a monotone transition system, which makes it amenable to efficient symbolic control synthesis techniques presented in (Saoud et al. (2019)).

There are few approaches in the literature that deal with data-driven symbolic control. In (Hashimoto et al. (2020)), an approach is presented for Lipschitz dynamical systems with partially unknown dynamics, the unknown dynamics being assumed to belong to some regression kernel. The approaches in (Milanese and Novara (2004); Sadraddini and Belta (2018)) also apply to Lipschitz dynamical systems with known bounds on the Lipschitz constants but use piecewise affine abstractions. In comparison, our approach does not assume that the system is Lipschitz but that it is monotone. While monotonicity is a strong property, monotone systems can be found in many practical applications such as adaptive cruise control (Saoud et al. (2019)),

⋆ This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 725144).

temperature regulation in buildings (Meyer et al. (2015)) or power networks (Zonetti et al. (2019)), to name a few.

In our previous work (Makdesi et al. (2021)), only the case without disturbance was considered, meanwhile, the approach presented in this paper can be applied to monotone systems with bounded disturbances. Moreover, the complexity of the approach presented in this paper is linear with respect to the number of samples of the dynamics and to the size of the partitions, when the complexity of the previous approach was polynomial of order $n + m$ where n and m are dimensions of state and input spaces.

The paper is organized as follows. In Section 2, we provide some preliminaries on monotone transition systems and their abstractions. In Section 3, we give a formal statement of the problem under consideration in the paper. Section 4 presents the main contributions of the paper by describing an efficient data-driven approach to abstraction of monotone systems with disturbances. Finally, Section 5 presents some numerical experiments to assess the effectiveness of our approach and to show how the amount of data and the choice of state and input space partitions affect the quality of the approximation.

2. PRELIMINARIES

In this section, after defining some notations, we introduce the necessary background on monotone transition systems.

2.1 Notations

$\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{N}$ denote the sets of real, non-negative real, and natural numbers, respectively. For a vector $x \in \mathbb{R}^n$, we denote by x^i its i-th coordinate, $i = 1, \dots, n$. We define a partial order $\preceq$ on $\mathbb{R}^n$ as follows: for $x, x' \in \mathbb{R}^n$ $x \preceq x'$ if and only if $x^i \leq x'^i$, for all $i = 1, \dots, n$. Similarly, $x \prec x'$ if and only if $x^i < x'^i$, for all $i = 1, \dots, n$. We denote $[x, x'] = \{y \in \mathbb{R}^n | \; x \preceq y \preceq x'\}$ and $[x, x') = \{y \in \mathbb{R}^n | \; x \preceq y \prec x'\}$. The empty set is denoted by $\emptyset$. Given a set X we denote 2^X to the set of subsets of X. A relation $R \subseteq X \times Y$ is identified with the set-valued map $R : X \to 2^Y$ where $R(x) = \{y \in Y | (x, y) \in R\}$.

2.2 Transition systems

Transition systems provide a unifying framework for considering continuous and discrete systems:

Definition 1. A *transition system* Σ is a tuple $\Sigma = (X, U, F, Y, H)$, where X is a set of states, U is a set of inputs, $F : X \times U \to 2^X$ is a transition relation, Y is a set of outputs, and $H : X \to Y$ is an output map.

Σ is finite if X and U are finite and it is said to be deterministic if for all $x \in X$ and all $u \in U$, $F(x, u)$ is singleton. We call any $x' \in F(x, u)$ a u-successor of the state x. In this work, we are assuming, for simplicity, that for all $x \in X$, for all $u \in U$, $F(x, u) \neq \emptyset$.

Alternating simulation relations make it possible to relate the behaviors of transition systems (see e.g. Tabuada (2009)). Let us consider two transition systems, $\Sigma_i = (X_i, U_i, F_i, Y_i, H_i)$, $i = 1, 2$ sharing the same sets of outputs ($Y_1 = Y_2 = Y$).

Definition 2. $R \subseteq X_1 \times X_2$ is an *alternating simulation relation* from Σ_1 to Σ_2 if the following conditions are satisfied:

- for all $x_1 \in X_1$, there exists $x_2 \in X_2$ such that $(x_1, x_2) \in R$;
- for all $(x_1, x_2) \in R$, $H_1(x_1) = H_2(x_2)$;
- for all $(x_1, x_2) \in R$, for all $u_1 \in U_1$, there exists $u_2 \in U_2$ such that for all $x_2' \in F_2(x_2, u_2)$, there exists $x_1' \in F_1(x_1, u_1)$ satisfying $(x_1', x_2') \in R$.

We say that Σ_1 is alternatingly simulated by Σ_2 if there exists an alternating simulation relation from Σ_1 to Σ_2. This is denoted by $\Sigma_1 \preceq_{AS} \Sigma_2$.

It can be shown (Tabuada (2009)) that if Σ_1 is alternatingly simulated by Σ_2, then any controller for Σ_1 can be refined to a controller for Σ_2 so that the closed loop output behavior of Σ_2 is included in that of Σ_1. This means that a controller synthesis problem with a specification on the output space for Σ_2 can be solved by considering the same problem for Σ_1.

2.3 Monotone systems

We now introduce the class of monotone transition systems. Let us consider the transition system $\Sigma = (X, U, F, Y, H)$, where $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$. Considering the partial order $\preceq$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, we define monotone systems as follow:

Definition 3. $\Sigma = (X, U, F, Y, H)$ is a *monotone transition system* if for all $x, x' \in X$, $u, u' \in U$ with $x \preceq x'$, $u \preceq u'$,

$$\forall y \in F(x, u), \; \exists y' \in F(x', u'), \; y \preceq y', \; \text{ and}$$
$$\forall y' \in F(x', u'), \; \exists y \in F(x, u), \; y \preceq y'.$$

def2.pdf

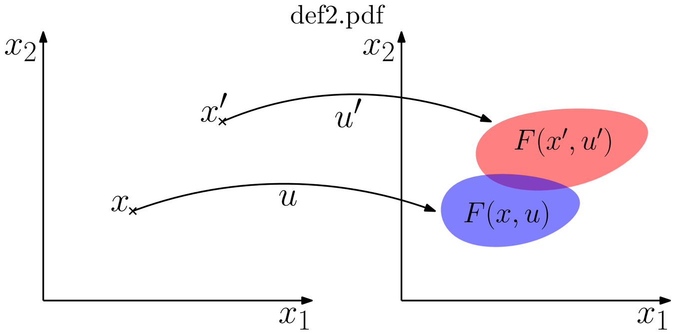

Fig. 1. Example of transitions of a monotone transition system.

Figure 1 shows the transition relations of two pairs of states and inputs. We can see that under our definition for the monotonicity, we do not require all the u'-successor of x' to be bigger than u-successor of x. In the case of deterministic systems, the equation in Definition 3 becomes:

$$x \preceq x', u \preceq u' \implies F(x, u) \preceq F(x', u') \tag{1}$$

which coincides with the classical definition of monotone systems (see e.g. Coogan and Arcak (2017)).

3. PROBLEM FORMULATION

Let us consider a transition system $\Sigma = (X, U, F, Y, H)$, defined on sets of states $X = \mathbb{R}^n$ and inputs $U = \mathbb{R}^m$ and whose dynamics is given by the set-valued map $F : X \times U \to 2^X$.

$$F(x,u) = \mathcal{F}(x,u) + W,\ x \in X, u \in U \qquad (2)$$

where $\mathcal{F} : X \times U \to X$ is an unknown monotone deterministic map (i.e. satisfying (1)), and $W \subseteq \mathbb{R}^n$ is a bounded set of disturbances with known lower and upper bounds $\underline{w}, \overline{w} \in \mathbb{R}^n$, i.e. $W = [\underline{w}, \overline{w}]$. It is straightforward to show that Σ is a monotone transition system. The output set Y and the output map H will be specified later.

In the following, we consider the system Σ as a black box, and we only have access to data generated by the system. Let us assume that we are given data consisting of a set of transitions

$$\mathcal{D} = \{(x_k, u_k, y_k) \,|\, y_k \in F(x_k, u_k), k = 1 \dots K\}.$$

The main problem considered in this paper is that of computing from the data $\mathcal{D}$, a symbolic model $\Sigma_{\mathcal{D}}$ that is alternatingly simulated by Σ. Indeed, as mentioned previously, that would make it possible to use $\Sigma_{\mathcal{D}}$ to synthesize a controller for Σ.

For $i = 1, \dots, n$, $j = 1, \dots, m$, let be given finite partitions $(\chi^i_{q^i})_{q^i \in Q^i}$ and $(\nu^j_{p^j})_{p^j \in P^j}$ of $\mathbb{R}$ where $Q^i = \{0, \dots, N^i\}$, $P^j = \{0, \dots, M^j\}$ and

$$\begin{cases} \chi^i_0 &= (-\infty, a^i_1) \\ \chi^i_{q^i} &= [a^i_{q^i}, a^i_{q^i+1}),\ q^i = 1, \dots, N^i - 1 \\ \chi^i_{N^i} &= [a^i_{N^i}, +\infty) \end{cases}$$

$$\begin{cases} \nu^j_0 &= (-\infty, b^j_1) \\ \nu^j_{p^j} &= [b^j_{p^j}, b^j_{p^j+1}),\ p^j = 1, \dots, M^j - 1 \\ \nu^j_{M^j} &= [b^j_{M^j}, +\infty) \end{cases}$$

where $a^i_1 < \dots < a^i_{N^i}$ and $b^j_1 < \dots < b^j_{M^j}$. Let us define $Q = Q^1 \times \dots \times Q^n$ and $P = P^1 \times \dots \times P^m$ and consider the finite rectangular partitions $(X_q)_{q \in Q}$, $(U_p)_{p \in P}$ of X and U, given for $q = (q^1, \dots, q^n)$ and $p = (p^1, \dots, p^m)$ by

$$X_q = \chi^1_{q^1} \times \dots \times \chi^n_{q^n},\ U_p = \nu^1_{p^1} \times \dots \times \nu^m_{p^m}.$$

We denote by $\underline{x}_q$, $\overline{x}_q$, $\underline{u}_p$, $\overline{u}_p$ the infimum and supremum, with respect to partial order $\preceq$, of X_q and U_p. Note that some components may be infinite if the intervals are unbounded.

We now define the set of outputs of Σ as $Y = Q$ and its output map as H given by $H(x) = q$ if and only if $x \in X_q$.

We now provide a formal statement of our problem:

Problem 1. Given disturbance lower and upper bounds $\underline{W}$, $\overline{W}$, the data set $\mathcal{D}$ and partitions $(X_q)_{q \in Q}$, $(U_p)_{p \in P}$, compute a symbolic model $\Sigma_{\mathcal{D}} = (Q, P, F', Y, H')$ such that $\Sigma_{\mathcal{D}} \preceq_{AS} \Sigma$.

4. DATA DRIVEN SYMBOLIC MODEL

In this section, we provide a solution to Problem 1. Based on the data set and the given partitions, we are looking for a minimal over-approximation of the unknown system dynamics.

4.1 Over-approximation of the transition relation

We define a box-shaped map $G : Q \times P \to 2^X$ with an upper bound $\overline{G}$ and a lower bound $\underline{G}$: i.e. $G(q,p) = [\underline{G}(q,p), \overline{G}(q,p)]$, for all $q \in Q$, $p \in P$. $\overline{G}(q,p)$ is defined component-wise as follows for $i = 1, \dots, n$:

$$\overline{G}^i(q,p) = \begin{cases} \min\limits_{(x_k,u_k) \succeq (\overline{x}_q, \overline{u}_p)} y^i_k + \overline{w}^i - \underline{w}^i \\ \qquad\qquad\qquad \text{if } \exists (x_k, u_k) \succeq (\overline{x}_q, \overline{u}_p), \\ +\infty \ \text{ otherwise.} \end{cases} \qquad (3)$$

Similarly, $\underline{G}(q,p)$ is defined component-wise as follows for $i = 1, \dots, n$:

$$\underline{G}^i(q,p) = \begin{cases} \max\limits_{(x_k,u_k) \preceq (\underline{x}_q, \underline{u}_p)} y^i_k + \underline{w}^i - \overline{w}^i \\ \qquad\qquad\qquad \text{if } \exists (x_k, u_k) \preceq (\overline{x}_q, \overline{u}_p), \\ -\infty \ \text{ otherwise.} \end{cases} \qquad (4)$$

The following result shows that G provides an over-approximation of the unknown map F:

Proposition 4. For all $q \in Q, p \in P$, for all $x \in X_q$, $u \in U_p$,

$$F(x,u) \subseteq G(q,p).$$

Proof. Let us consider $q \in Q$, $p \in P$, $x \in X_q$, $u \in U_p$. If there exists $(x_k, u_k) \succeq (\overline{x}_q, \overline{u}_p)$, we have for all $i \in \{1, \dots, n\}$:

$$\mathcal{F}^i(\overline{x}_q, \overline{u}_p) + \overline{w}^i \leq \mathcal{F}^i(x_k, u_k) + \overline{w}^i$$

for all k such that $(x_k, u_k) \succeq (\overline{x}_q, \overline{u}_p)$. Therefore,

$$\mathcal{F}^i(\overline{x}_q, \overline{u}_p) + \overline{w}^i \leq \min_{(x_k,u_k) \succeq (\overline{x}_q, \overline{u}_p)} \mathcal{F}^i(x_k, u_k) + \overline{w}^i.$$

We also have, for all $k = 1, \dots, K$,

$$\mathcal{F}^i(x_k, u_k) + \underline{w}^i \leq y^i_k.$$

Hence,

$$\begin{aligned} \mathcal{F}^i(\overline{x}_q, \overline{u}_p) + \overline{w}^i &\leq \min_{(x_k,u_k) \succeq (\overline{x}_q, \overline{u}_p)} y^i_k + \overline{w}^i - \underline{w}^i \\ &\leq \overline{G}^i(q,p). \end{aligned}$$

We have also $\mathcal{F}^i(\overline{x}_q, \overline{u}_p) + \overline{w}^i \leq \overline{G}^i(q,p)$ when $\overline{G}^i(q,p) = +\infty$. In a similar way we can show that $\mathcal{F}^i(\underline{x}_q, \underline{u}_p) + \underline{w}^i \geq \underline{G}^i(q,p)$. Hence, $F(x,u) \subseteq G(q,p)$. □

Actually, we can claim a slightly stronger result than that of Proposition 4:

Claim 5. Let $\tilde{F} : X \times U \to 2^X$ be of the form (2) and such that $y_k \in \tilde{F}(x_k, u_k)$, for all $k = 1, \dots, K$, we have $\tilde{F}(x,u) \subseteq G(q,p)$, for all $q \in Q$, $p \in P$, for all $x \in X_q$, $u \in U_p$.

The proof of the previous claim is identical to that of Proposition 4 and is therefore omitted. The previous claim essentially states that G is an over-approximation of all maps of the form (2) that are consistent with the data. The following result states that in the absence of disturbances, it is actually the minimal over-approximation:

Proposition 6. Let $\underline{w} = \overline{w} = 0$, let $\tilde{G} : Q \times P \to 2^X$ be a box-shaped map such that the statement of Claim 5 holds. Then, we have for all $q \in Q$, $p \in P$, $G(q,p) \subseteq \tilde{G}(q,p)$.

Proof. Let us consider $q \in Q$, $p \in P$. From (3), we get that for all $(x_k, u_k) \succeq (\overline{x}_q, \overline{u}_p)$, $\mathcal{F}(x_k, u_k) \succeq \overline{G}(q,p)$. By

(3) and monotonicity of $\mathcal{F}$, we also have for all $(x_k, u_k) \preceq (\overline{x}_q, \overline{u}_p)$, $\mathcal{F}(x_k, u_k) \preceq \overline{G}(q, p)$. Then, we can choose a monotone map $\tilde{\mathcal{F}} : X \times U \times X$ such that $\tilde{\mathcal{F}}(x_k, u_k) = \mathcal{F}(x_k, u_k)$, for all $k = 1, \ldots, K$ and $\tilde{\mathcal{F}}(\overline{x}_q, \overline{u}_p) = \overline{G}(q, p)$. Since Claim 5 holds, it follows that $\tilde{\mathcal{F}}(\overline{x}_q, \overline{u}_p) \in \tilde{G}(q, p)$. Therefore, $\overline{G}(q, p) \in \tilde{G}(q, p)$. Using a similar approach, we can show that $\underline{G}(q, p) \in \tilde{G}(q, p)$. Since $\tilde{G}$ is a boxed-shaped map, we get $[\underline{G}(q, p), \overline{G}(q, p)] \subseteq \tilde{G}(q, p)$. □

The previous proposition shows that the conservatism of our approach is minimal in the absence of disturbances. In the presence of disturbances, the absence of information on the actual value of the disturbance at samples prevents us from achieving minimal conservatism.

4.2 Resulting symbolic model

We can now define the following transition system $\Sigma_{\mathcal{D}} = (Q, P, F', Y, H')$ where for all $q \in Q$, $p \in P$, $F'(q, p) = \{q' \in Q | \ X_{q'} \cap G(q, p) \neq \emptyset\}$ and $H'(q) = q$.

Let us remark that, formally Q and P are finite subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ so they are equipped with partial orders. Let us state the main result of the paper:

Theorem 7. $\Sigma_{\mathcal{D}} \preceq_{AS} \Sigma$ and the alternating simulation relation is given by

$$R = \{(x, q) \in X \times Q | \ x \in X_q\}.$$

Moreover, the symbolic model $\Sigma_{\mathcal{D}}$ is a monotone transition system.

Proof. Let us start by the alternating simulation relation. It is obvious from the definitions of R, H, H' that the first two conditions of the alternating simulation relation are satisfied. For the last condition, we have from Proposition 4, $F(x, u) \subseteq G(q, p)$ for all $q \in Q$, $p \in P$, for all $x \in X_q$, $u \in U_p$. Therefore, for all $(x, q) \in R$, for all $p \in P$, there exists $u \in U_p$ such that for all $x' \in F(x, u)$, where $x' \in X_{q'}$, $x' \in G(q, p)$ implies that we have $q' \in F'(q, p)$. From the definition of x', q' we have $(x', q') \in R$.

We now prove the monotonicity of the symbolic model. Let $q_1, q_2 \in Q$, $p_1, p_2 \in P$ with $q_1 \preceq q_2$ and $p_1 \preceq p_2$. Let us consider the set of points

$$K^+(q_i, p_i) = \{(x_k, u_k) \mid (x_k, u_k) \succeq (\overline{x}_{q_i}, \overline{u}_{p_i})\}, \ i = 1, 2.$$

Let us remark that $(q_1, p_1) \preceq (q_2, p_2)$ implies that $(\overline{x}_{q_1}, \overline{u}_{p_1}) \preceq (\overline{x}_{q_2}, \overline{u}_{p_2})$, which in turn implies that we have $K^+(q_2, p_2) \subseteq K^+(q_1, p_1)$. It follows from (3) that

$$\overline{G}(q_1, p_1) \preceq \overline{G}(q_2, p_2). \tag{5}$$

Equation (4), with the same reasoning, leads to

$$\underline{G}(q_1, p_1) \preceq \underline{G}(q_2, p_2). \tag{6}$$

Then, let $q'_1 \in F'(q_1, p_1)$, we have that $q'_1 \preceq \overline{q}'_1$, where $\overline{q}'_1$ is such that $\overline{G}(q_1, p_1) \in X_{\overline{q}'_1}$. Let q'_2 be such that $\overline{G}(q_2, p_2) \in X_{q'_2}$, then from (5), it follows that $\overline{q}'_1 \preceq q'_2$, and therefore that $q'_1 \in\preceq q'_2$. Also, by definition of F', we get that $q'_2 \in F'(q_2, p_2)$. Along the same lines and using (6), we can prove that for all $q'_2 \in F'(q_2, p_2)$, there exists $q'_1 \in F'(q_1, p_1)$ such that $q'_1 \in\preceq q'_2$. Hence, the transition system $\Sigma_{\mathcal{D}}$ is monotone. □

Knowing that $\Sigma_{\mathcal{D}}$ is a monotone system, makes it possible to use efficient discrete controller synthesis techniques (Saoud et al. (2019)).

We would like to end the section with a discussion regarding the computational complexity of the approach. The computational cost of computing the symbolic model is dominated by that of computing the functions $\overline{G}$ and $\underline{G}$. Using equations (3) and (4), it appears that this can be done with a complexity in $\mathcal{O}(N \times |Q| \times |P|)$, i.e. linear with respect to the number of data samples and also with respect to the size of the partition.

5. NUMERICAL EXAMPLE

In this section, we report some numerical experiments to show the performance of our approach.

Let us consider a model with two vehicles moving in one lane on an infinite straight road. The leader is uncontrollable (vehicle 2) while the follower is controllable (vehicle 1). A discrete-time approximation of this model is given by equations:

$$\begin{aligned} d(k+1) &= d(k) + (v_1(k) - v_2(k))T_0 + w^1(k) \\ v_1(k+1) &= v_1(k) + \alpha(u(k), v_1(k))T_0 + w^2(k) \\ v_2(k+1) &= v_2(k) + w^3(k) \end{aligned} \tag{7}$$

Here $u(k)$ is the control input which is the torque applied to the wheels, and $d(k)$ is the signed distance between the vehicles. $w(k) = [w^1(k), w^2(k), w^3(k)]$ are disturbances which belong to a rectangular set W with known lower and upper bounds. α is a nonlinear function given by

$$\alpha(u, v) = u - M^{-1}(f_0 + f_1 v + f_2 v^2).$$

The vector of parameters $f = (f_0, f_1, f_2) \in \mathbb{R}^3_+$ describes road friction and vehicle aerodynamics whose numerical values are taken from (Nilsson et al. (2015)): $f_0 = 51\ N, f_1 = 1.2567\ N\ s/m,\ f_2 = 0.4342 N\ s^2/m^2$. For the rest of parameters, we chose $M = 1370\ kg, T_0 = 0.5\ s$.

The system can be made monotone by making the change of variable $v_2(k) = -v_2(k)$. We only use this model to generate the random set of data points which then is used by the algorithm to calculate the symbolic abstraction.

We sample the component of the data points, input and disturbance from uniform distributions in the sets

$$\begin{aligned} X_{\mathcal{D}} &= [-100, 0] \times [10, 30] \times [-22, -18] \\ U_{\mathcal{D}} &= [-3, 3] \\ W_{\mathcal{D}} &= [-0.1, 0.1] \times [-0.1, 0.1] \times [-0.1, 0.1] \end{aligned}$$

We define the partitions for each coordinate of the state space as follow:

$$(-\infty, a^i_1), [a^i_1, \frac{1}{N^i}(a^i_{N^i} - a^i_1) + a^i_1), \ldots,$$
$$[\frac{N^i - 1}{N^i}(a^i_{N^i} - a^i_1) + a^i_1, a^i_{N^i}), [a^i_{N^i}, +\infty)$$

with $[a^1_1, a^1_{N^1}] \times [a^2_1, a^2_{N^2}] \times [a^3_1, a^3_{N^3}] \subset X_{\mathcal{D}}$. In a similar way we define the partitions for u:

$$(-\infty, b_1), [b_1, \frac{1}{M}(b_M - b_1) + b_1), \ldots,$$
$$[\frac{M - 1}{M}(b_M - b_1) + b_1, b_M), [b_M, +\infty)$$

with $[b_1, b_M] \subset U_{\mathcal{D}}$.

In order to measure the conservatism in our abstraction we introduce the performance criterion

$$\mu(\mathcal{D},Q,P)=\frac{\sum_{q,p}\left(vol(X_q\times U_p)\times\sum_{q'\in F'(q,p)}vol(X_{q'})\right)}{\sum_{q,p}\left(vol(X_q\times U_p)\right)vol(W_{\mathcal{D}})}$$
$$\forall q\in Q,\forall q'\in Q,\forall p\in P \text{ such that}$$
$$X_q\subseteq X_{\mathcal{D}},X'_q\subseteq X_{\mathcal{D}},U_p\subseteq U_{\mathcal{D}}$$

Intuitively, $\mu(\mathcal{D},Q,P)$ measures the degree of conservatism of the abstraction. The denominator represents the volume of the unknown map F for the part of space where we can find an approximation, whereas the numerator represents the volume of our over-approximation. μ can take its value in the interval $[1,\infty)$ and the the smaller its value is, the more accurate the abstraction.

5.1 Abstraction

For our first experiment, we fixed the partitions and computed the abstraction using different numbers of data points. For a given number of data points, we sampled five different random sets of data, and calculated the associated abstraction. For the partitions, we chose $N^1=30$, $N^2=30$, $N^3=20$, $M=12$, $a_1^1=-80$, $a_{N^1}^1=-20$, $a_1^2=15$, $a_{N^2}^2=25$, $a_1^3=-21.5$, $a_{N^3}^3=-18.5, b_1=-2.5$, $b_M=2.5$. In Figure 2, the average times needed to compute the abstraction for the five sets of samples are plotted with respect to the number of data points. We can see that the computation time increases linearly with the number of data points. In Figure 3, the average values of μ are plotted with respect to the number of data points. We can see how increasing the number of data points can reduce the conservatism of our abstraction.

For the second experiment, we fixed the data set $\mathcal{D}$, but changed the size of the partitions. For each instance, we increased all the values of N^1,N^2,N^3,M and calculated the abstraction. We chose a number of data points $N=10^4$, and we fixed the values $a_1^1=-80$, $a_{N^1}^1=-20$, $a_1^2=15$, $a_{N^2}^2=25$, $a_1^3=-21.5$, $a_{N^3}^3=-18.5, b_1=-2.5$, $b_M=2.5$. In Figure 4, we can see how the time of execution increases linearly with the size of partitions. Figure 5 shows how increasing the number of partition elements also improves the abstraction.

5.2 Controller synthesis

We briefly show that our abstractions are suitable for controller synthesis purposes. Since the leader vehicle is

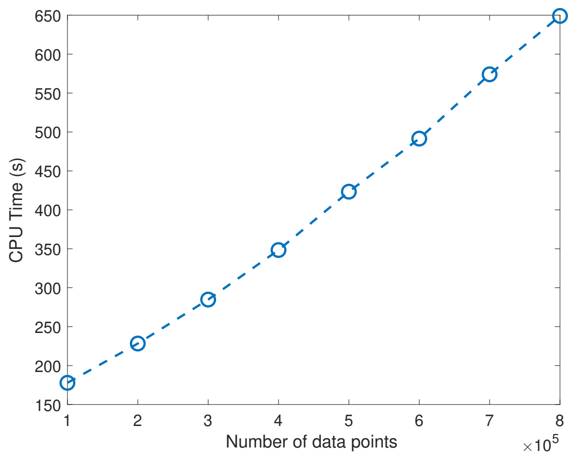

Fig. 2. Average time of execution with respect to the number of data points

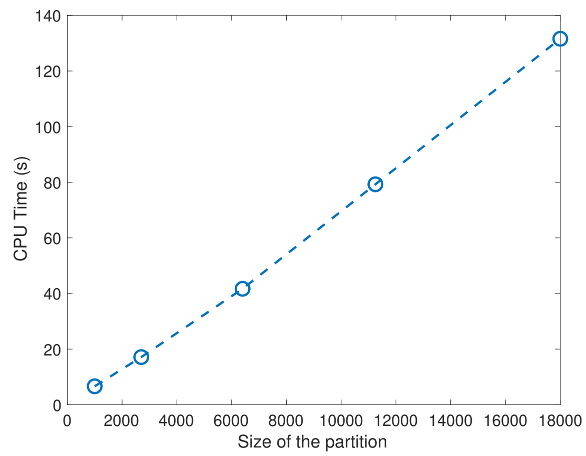

Fig. 4. Time of execution with respect to the number of elements of the partitions

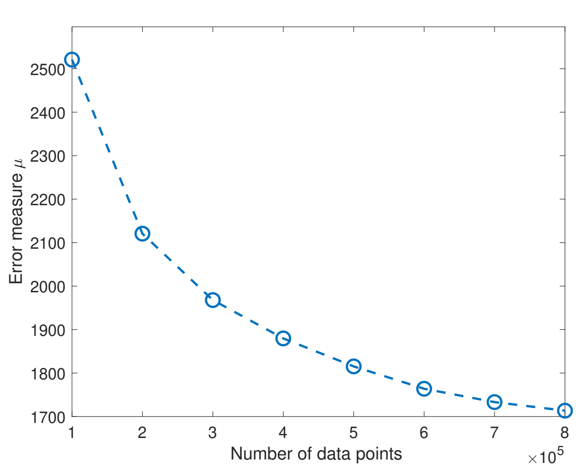

Fig. 3. Average value of $\mu(\mathcal{D},Q,P)$ with respect to the number of data points

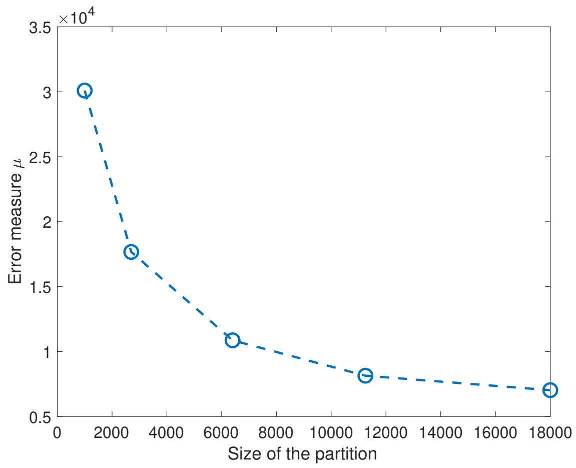

Fig. 5. $\mu(\mathcal{D},Q,P)$ with respect to the number of elements of the partitions

uncontrollable, it is natural to consider a specification given under the form of the following assume-guarantee contract:

$$\forall k \in \mathbb{N}\ v_2(k) \in [\underline{v}_2, \overline{v}_2] \implies \forall k \in \mathbb{N},\ v_1(k) \in [\underline{v}_1, \overline{v}_1] \wedge d(k) \in [\underline{d}, \overline{d}]. \tag{8}$$

Essentially, the specification states that if the velocity of the leader remains within bounds $[\underline{v}_2, \overline{v}_2]$, then the velocity of the follower and the distance remain within bounds $[\underline{v}_1, \overline{v}_1]$ and $[\underline{d}, \overline{d}]$, respectively. In the following, we consider the numerical values $\underline{d} = -80$, $\overline{d} = -20$, $\underline{v}_1 = 15$, $\overline{v}_1 = 25$, $\underline{v}_2 = -21.5$, $\overline{v}_2 = -18.5$.

The synthesis of symbolic controllers enforcing assume-guarantee contracts such as (8) has been considered in (Saoud et al. (2020)) and we use similar techniques in the present case. The abstraction was computed based on a data set of 10^6 samples with the same partition as that reported in the first experiment above. The controllable set of the resulting symbolic controller is shown on Figure 6.

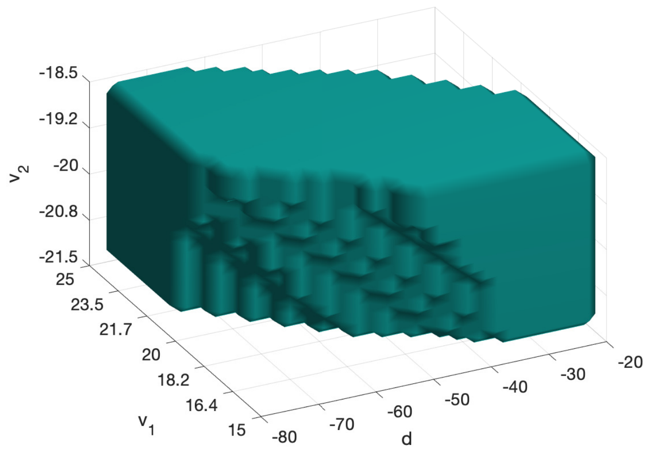

Fig. 6. Controllable set of the symbolic controller enforcing the assume-guarantee contract (8)

6. CONCLUSION

In this paper, we presented a data-driven approach for computing abstractions of monotone systems. The approach is computationally tractable and the abstractions can be used to design controllers that are correct by design. In the future, we would like to develop data-driven approaches to estimate the bounds on the disturbances, which currently are assumed to be known. We also aim at developing similar approaches for other classes of systems, such as e.g. Lispchitz continuous systems. We would also like to extend our approach to refine symbolic models and controllers online while collecting samples of closed loop behaviors.

REFERENCES

Belta, C., Yordanov, B., and Gol, E.A. (2017). *Formal methods for discrete-time dynamical systems*, volume 89. Springer.

Bloem, R., Jobstmann, B., Piterman, N., Pnueli, A., and Sa'ar, Y. (2012). Synthesis of reactive (1) designs. *Journal of Computer and System Sciences*, 78(3), 911–938.

Coogan, S. and Arcak, M. (2017). Finite abstraction of mixed monotone systems with discrete and continuous inputs. *Nonlinear Analysis: Hybrid Systems*, 23, 254–271.

Girard, A. (2012). Controller synthesis for safety and reachability via approximate bisimulation. *Automatica*, 48(5), 947–953.

Hashimoto, K., Saoud, A., Kishida, M., Ushio, T., and Dimarogonas, D. (2020). Learning-based safe symbolic abstractions for nonlinear control systems. *arXiv preprint arXiv:2004.01879*.

Makdesi, A., Girard, A., and Fribourg, L. (2021). Data-driven abstraction of monotone systems. *Proceedings of the 3rd Conference on Learning for Dynamics and Control*.

Meyer, P.J., Girard, A., and Witrant, E. (2015). Safety control with performance guarantees of cooperative systems using compositional abstractions. *IFAC-PapersOnLine*, 48(27), 317–322.

Milanese, M. and Novara, C. (2004). Set membership identification of nonlinear systems. *Automatica*, 40(6), 957–975.

Nilsson, P., Hussien, O., Balkan, A., Chen, Y., Ames, A.D., Grizzle, J.W., Ozay, N., Peng, H., and Tabuada, P. (2015). Correct-by-construction adaptive cruise control: Two approaches. *IEEE Transactions on Control Systems Technology*, 24(4), 1294–1307.

Pola, G. and Di Benedetto, M.D. (2019). Control of cyber-physical-systems with logic specifications: a formal methods approach. *Annual Reviews in Control*, 47, 178–192.

Ramadge, P.J. and Wonham, W.M. (1987). Supervisory control of a class of discrete event processes. *SIAM journal on control and optimization*, 25(1), 206–230.

Reissig, G., Weber, A., and Rungger, M. (2016). Feedback refinement relations for the synthesis of symbolic controllers. *IEEE Transactions on Automatic Control*, 62(4), 1781–1796.

Sadraddini, S. and Belta, C. (2018). Formal guarantees in data-driven model identification and control synthesis. In *Proceedings of the 21st International Conference on Hybrid Systems: Computation and Control (part of CPS Week)*, 147–156.

Saoud, A., Girard, A., and Fribourg, L. (2020). Contract-based design of symbolic controllers for safety in distributed multiperiodic sampled-data systems. *IEEE Transactions on Automatic Control*.

Saoud, A., Ivanova, E., and Girard, A. (2019). Efficient synthesis for monotone transition systems and directed safety specifications. In *IEEE 58th Conference on Decision and Control*, 6255–6260.

Tabuada, P. (2009). *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media.

Zonetti, D., Saoud, A., Girard, A., and Fribourg, L. (2019). Decentralized monotonicity-based voltage control of dc microgrids with zip loads. *IFAC-PapersOnLine*, 52(20), 139–144.